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Induced Traces on Coaction Crossed Product C^*-algebras

TSUYOSHI KAJIWARA
Department of Mathematics, College of Liberal Arts and Sciences
Okayama University

0. Introduction

Trace is one of the most fundamental objects in representation theory of C*-algebra. In [3] we characterized induced traces on group extensions and C*-crossed products by using dual of action. Because of noncommutativity of groups, we have used coaction in C*-algebra sense. In this paper, contrary we consider traces on a C*-coaction crossed product $A \times_\delta G$ induced from a trace on the basis C*-algebra A in the sense of coaction.

The result is as follows. Lower semicontinuous semi-finite trace on C*-cocrossed product $A \times_\delta G$ is induced from a trace on A if this trace is relatively invariant with respect to the modular function Δ_G of G under the canonical action $\hat{\delta}$ dual to the original coaction δ. In the case where G is abelian, this result is proved in [6].

1. Result

Let A be a separable C^*-algebra and G be a 2'nd countable amenable locally compact group. Coacion of G on a C*-algebra A means an injective $*$-homomorphism δ from A to the multiplier algebra $M(A \otimes \mathrm{C}^*_r(G))$ of the spatial tensor product $A \otimes \mathrm{C}^*_r(G)$ such that $(\delta \otimes i)\delta = (i \otimes \delta_G)\delta$ holds. The notation δ_G means the canonical comultiplication on $\mathrm{C}^*_r(G)$.

Suppose that a coaction δ of G on A is given. Coaction C*-crossed product (C*-cocrossed product) $A \times_\delta G$ is the C*-algebra generated by $\{\delta(A), C_0(G)\}$ on $\mathcal{H} \otimes L^2(G)$ where $\mathcal{H}$ is the representation space of A. We put this C*-algebra B. We refer basic properties of C*-coaction and C*-cocrossed products to [2]. Canonical dual action $\hat{\delta}$ of G on B is given by the adjoint of right regular representation of G on $\mathcal{H} \otimes L^2(G)$. By the definition, this action constitutes a system of imprimitivity with a copy of $\mathrm{C}_0(G)$ in B.

Let ϕ be a lower semicontinuous semifinite (l.s.s.) trace on B. We consider a condition under which this trace ϕ is given by a trace ψ on A in a canonical way. We call ϕ is induced from (δ, j)-invariant trace ψ on A if δ can extend to a coaction $\tilde{\delta}$ on weak closure of $\pi_\psi(A)$, and $\pi_\phi(B)''$ is the W*-cocrossed product $(\pi_\phi(A))'' \times_{\tilde{\delta}} G$ and ϕ is the restriction of induced trace on $(\pi_\phi(A))'' \times_{\tilde{\delta}} G$ to A. We refer the induction of traces in the situation of group action and coaction to [3], and refer the definition of (δ, j)-invariance of traces to [7] and don't state the detail, because we don't use the detail of this definition.

We denote the modular function of G by Δ_G. Let ϕ be a l.s.s. trace on B satisfying the following property.

$$\phi(\hat{\delta}(g)(x)) = \Delta_G(g)^{-1}\phi(x) \qquad \forall x \in B^+,\ \forall g \in G$$

We call such a ϕ Δ_G-relatively invariant under $\hat{\delta}$.

An ideal J in A is called δ-invariant if $\delta(J) \subset M(J \otimes \mathrm{C}^*(G))$ [1], [4]. Since G is assumed to be amenable, there is no problem concerning the definition of invariance of ideals. If J is δ invariant, we can consider coaction δ_J (restriction to J) and quotient coaction δ^J on A/J. Put π_ϕ be a GNS representation of B given by ϕ.

LEMMA 1. *There exists a δ invariant ideal J in A and $\pi_\phi(B)$ is of the form $(A/J) \times_{\delta^J} G$.*

Since traces on $A \times_\delta G/J \times_{\delta_J} G$ (resp. A/J) can be considered traces on $A \times_{\delta^J} G$ (resp. A), we can assume that ϕ is faithful. We identify A with $\pi_\phi(A)$. M be a W^*-closure of $\pi_\phi(B)$.

LEMMA 2.

There exist a von Neumann subalgebra N in M and coaction $\tilde{\delta}$ of G on N such that M is isomorphic to W^*-cocrossed coproduct $N \times_{\tilde{\delta}} G$. C-algebra A is contained in N and σ weakly dense in N. Moreover, $\tilde{\tilde{\delta}}$ is the dual action $\hat{\tilde{\delta}}$ of $\tilde{\delta}$.*

For $x \in M^+$, put $E^\delta(x) = \int_G \tilde{\delta}_g(x)\, dg$. Then E^δ is an operator valued weight from M to N.

LEMMA 3.

There exists a unique a $(\tilde{\delta}, j)$-invariant n.s.f trace $\tilde{\psi}$ on N such that $\tilde{\phi}$ is of the form $\tilde{\psi} \circ E^{\tilde{\delta}}$.

LEMMA 4.

The restriction ψ of $\tilde{\psi}$ to A is semifinite. If ϕ is moreover densely defined, ψ is also densely defined.

THEOREM.

Let ϕ be a lower semicontinuous semifinite trace on a B. If and only if ϕ is Δ_G-relatively invariant under the canonical dual action of G. Then ϕ is induced from a lower semi continuous semifinite trace ψ on A.

REMARK 1.

The representation π_ϕ is induced from π_ψ in the sense of C*-*coaction. We refer the definition of induced representation of coaction to [1], [6] and we don't state the detail.*

REMARK 2.

*In the situation of coaction, we don't have convenient Banach *- algebra, so the formulation of induction of traces in purely* C*-*algebraic sense is not clear to us.*

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